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A Conservation Law Related to Kelvin's Circulation Theorem

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The governing equations for a moving hydrodynamic surface lead to a local conservation law for a surface velocity variable q_s not in common use. When the surface is closed and applied forces are conservative, the law reduces to Kelvin's circulation theorem. When the flow is irrotational, it reduces to Bernoulli's law. Incorporation of the conservation law into a numerical water wave model cast in an Eulerian representation can result in (1) reduction of the prognostic equations from two spatial dimensions to one, and (2) realization of formal accuracy to all orders in nonlinearity. In the companion paper [1], the shallow water diagnostic equation (Poisson's equation) is also reduced to a one-dimensional problem. The prognostic equation derived here thus allows a purely one-dimensional treatment of traditionally two-dimensional shallow water waves. This yields significant resolution and execution speed benefits for the numerical integration of the overall system. Techniques also exist that reduce the deep water diagnostic equation to one dimension. Thus the new prognostic equation should be useful in modeling two-dimensional deep water waves as a one-dimensional problem.

1. Introduction

This brief report derives a result used by one of the authors (J.M.W.) in construction of a unified model of shallow water waves [1]. In an earlier investigation, one of the authors (B.E.M.) came across an unfamiliar "conservation-of-velocity" law related to Kelvin's circulation theorem. Together [2] we recognized the result as the exact counterpart of an approximate expression used by Witting [3] to derive reflection coefficients for long linear waves. For irrotational flows, the conservation of velocity law is a variant of Bernoulli's law, but the general form holds for rotational flows as well.

In the next section we derive the conservation law, demonstrate its connection with the circulation theorem and with Bernoulli's law, and point out its utility in three

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equation wave models. Such models have as dependent variables u_s, v_s, and h, denoting, respectively, horizontal and vertical surface velocity components and wave height. The governing equations consist of two prognostic equations for time advancement of h and a surface velocity variable q_s to be defined in (9) below. The system is closed by a diagnostic relation derived from Poisson's equation taking into account appropriate boundary conditions. The form of the diagnostic relation may be application dependent (e.g., deep versus shallow water waves) and will not be considered in detail here. In the companion paper [1], the author uses a power series technique to relate surface and bottom velocity potentials for shallow water waves. For deep water waves, the diagnostic relation can involve a singular integral equation [4] or a mapping which renders the surface flat [5].

2. Surface Equations

A. *Derivation of the Conservation-of-Velocity Law*

We choose a coordinate system with x horizontal, y vertically upward, and $h(x, t)$ the elevation of the surface (or, more generally, an arbitrary continuous surface of markers moving with the fluid). The equations governing the fluid and surface are

$$\frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = v_s \tag{1}$$

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho}\left(F_x - \frac{\partial p}{\partial x}\right) \tag{2}$$

$$\frac{Dv}{Dt} \equiv \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\rho}\left(F_y - \frac{\partial p}{\partial y}\right) - g, \tag{3}$$

where subscript s denotes evaluation at the surface, ρ is the fluid density, p the pressure, g the acceleration of gravity, and (F_x, F_y) are horizontal and vertical components of remaining forces (e.g., viscosity, turbulent Reynolds stress, or external body forces).

The following relations are of assistance in dealing with surface quantities. Let $Q(x, y, t)$ denote a variable of interest (specifically, u or v). Then

$$Q_s(x, t) \equiv Q(x, h(x, t), t), \tag{4}$$

so that

$$\begin{aligned} \frac{\partial Q_s}{\partial t} &= \left(\frac{\partial Q}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial Q}{\partial y}\right)_s \\ &= \left(\frac{\partial Q}{\partial t} + (v_s - u_s h')\frac{\partial Q}{\partial y}\right)_s, \end{aligned} \tag{5}$$

where the second equality follows from (1). Also,

$$\frac{\partial Q_s}{\partial x} = \left(\frac{\partial Q}{\partial x} + h' \frac{\partial Q}{\partial y}\right)_s, \tag{6}$$

where

$$h' \equiv \frac{\partial h}{\partial x}. \tag{7}$$

It is of interest to note in (5) that the time derivative at fixed x moving vertically with the surface is *not* the material derivative D/Dt in (2) and (3) with the horizontal advection dropped. One finds, however, from (5) and (6) that

$$\frac{\partial Q_s}{\partial t} + u_s \frac{\partial Q_s}{\partial x} = \left(\frac{\partial Q}{\partial t} + u \frac{\partial Q}{\partial x} + v \frac{\partial Q}{\partial y}\right)_s, \tag{8}$$

which *is* the material derivative at the surface. We now define a variable

$$q_s(x, t) \equiv u_s + h' v_s, \tag{9}$$

which may be recognized as the x derivative of the line integral $\int \mathbf{v} \cdot \mathbf{dl}$ taken along the surface. (This line integral when taken over the extent of a solitary wave has been referred to as the circulation of the wave [6].) Taking the time derivative of (9) and substituting $(Du/Dt)_s - u_s(\partial u_s/\partial x)$ for $\partial u_s/\partial t$ and likewise for v_s leads, with (1)–(3), to the following:

$$\frac{\partial q_s}{\partial t} = \frac{1}{\rho}(F_x + h'F_y) - \frac{\partial}{\partial x}\left(\frac{p_s}{\rho} + gh - \frac{u_s^2 + v_s^2}{2} + u_s q_s\right). \tag{10}$$

This result does not require the flow to be irrotational. If the fluid is barotropic ($\rho = \rho(p)$) then one replaces p_s/ρ in (10) with $(\int dp/\rho)_s$. Equation (10) is the result we refer to as the "conservation-of-velocity law," since in case $F_x = F_y = 0$ it leads to constancy of $\int q_s \, dx$ for flows which are periodic or quiescent at infinity.

Equation (10) has the attractive property that for $\mathbf{F} = 0$ it allows prediction of q_s from quantities known only at the surface. The same holds for prediction of h from Eq. (1). Thus for the three quantities u_s, v_s, h we have two prognostic equations involving differentiation only along the surface. This is a significant result for modeling purposes, since it reduces the prognostic equations from two dimensions to one. A similar equation structure has been exploited successfully using Bernoulli's law for the case of irrotational flow to simulate large-amplitude deep water waves [4]. One would expect, therefore, that Eq. (10) would be of value in the simulation of forced waves or in situations where an alternative to the Bernoulli formulation may be desirable.

B. *Relation to the Circulation Theorem*

Equation (10) becomes recognizable as a variant of Kelvin's circulation theorem cast in an Eulerian representation upon consideration of the following. Let a and b be abscissae of two points on the surface and moving with the fluid. Then

$$\frac{D}{Dt}\int_a^b q_s\,dx = \int_a^b \frac{\partial q_s}{\partial t}\,dx + u_s q_s \Big|_a^b$$
$$= \int_a^b \left(\frac{\partial q_s}{\partial t} + \frac{\partial u_s q_s}{\partial x}\right) dx. \tag{11}$$

On the surface, $dy = h'\,dx$, so that (9)–(11) give

$$\frac{D}{Dt}\int_a^b u_s\,dx + v_s\,dy = \int_a^b \frac{1}{\rho}(F_x\,dx + F_y\,dy)$$
$$+ \left(\frac{u^2+v^2}{2} - \frac{p_s}{\rho} - gh\right)\Big|_a^b. \tag{12}$$

For conservative forcing, this becomes a familiar result of Lamb [7]. If the surface closes upon itself, then a and b are the same point and conservative forcing leads to constancy of the circulation in a moving circuit. Thus the circulation theorem is recovered.

C. *Relation to Bernoulli's law*

For irrotational flows, Bernoulli's law is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}(u^2+v^2) - \frac{p}{\rho} - gy + f(t), \tag{13}$$

where $\phi(x, y, t)$ is the velocity potential and f an arbitrary function of time. We note from (6) and (9) that

$$q_s = \frac{\partial \phi_s}{\partial x}. \tag{14}$$

The assumption of irrotational flow requires the forcing term in (10) to be the gradient of a potential $W(x, y, t)$. Then an x integral of (10) yields, with (13) and (14),

$$\frac{\partial \phi_s}{\partial t} = \frac{1}{\rho} W_s - \frac{p_s}{\rho} - gh + \frac{u_s^2 + v_s^2}{2} - u_s \frac{\partial \phi_s}{\partial x} + f(t). \tag{15}$$

Taking the next to last term to the left-hand side and recognizing the result as $D\phi_s/Dt$, then subtracting $\mathbf{v}\cdot\nabla\mathbf{v}$ from both sides, we find

$$\left(\frac{\partial\phi}{\partial t}\right)_s = \frac{1}{\rho}W_s - \frac{p_s}{\rho} - gh - \frac{u_s^2 + v_s^2}{2} + f(t). \tag{16}$$

This is a simple generalization of (13) for conservative forcing, so that (10) reduces to Bernoulli's law for irrotational flow.

3. Application to Wave Models

Assuming the surface forcing and pressure to be known, Eqs. (1) and (10) provide two constraints for the three variables h, u_s, and v_s. The third constraint necessary to close the system is a two-dimensional boundary value problem resulting from incompressibility. Equations (1) and (10) involve only surface values as a function of x and t. There is thus strong motivation to reduce the two-dimensional (x, y) boundary value problem to one dimension (x). If this can be done, the entire system becomes one dimensional in space, yielding substantial advantages in available resolution and speed for a numerical model of system. For shallow water waves this can be accomplished through expansion of the velocity potential in powers of h [1]. For deep water waves one can choose between a singular integral equation method [4] or a conformal mapping to a flat surface [5]. We will simply outline the problem and its boundary conditions. The mechanics of actually solving the boundary value problem as a one-dimensional system are addressed elsewhere [1, 4, 5].

The following discussion will be aided by definition of unit vectors tangent to and normal to the surface:

$$\boldsymbol{\tau} = (1, h')/(1 + h'^2)^{1/2} \tag{17}$$

$$\mathbf{n} = (-h', 1)/(1 + h'^2)^{1/2}. \tag{18}$$

We consider two general cases: the velocity potential formulation for irrotational flows and the streamfunction formulation for rotational flows. For irrotational flows we have

$$u = \frac{\partial\phi}{\partial x} \tag{19}$$

$$v = \frac{\partial\phi}{\partial y} \tag{20}$$

and

$$\nabla^2\phi = 0. \tag{21}$$

Integration of (10) forward in time yields q_s, which is related to the tangential derivative of ϕ by the following combination of (9) and (17):

$$q_s = (1 + h'^2)^{1/2} (\boldsymbol{\tau} \cdot \nabla\phi)_s. \tag{22}$$

On the other hand, time advancement of h requires knowledge of the normal derivative:

$$\frac{\partial h}{\partial t} = (1 + h'^2)^{1/2} (\mathbf{n} \cdot \nabla\phi)_s. \tag{23}$$

The boundary value problem for this case is then to find $\partial\phi/\partial n$ given $\partial\phi/\partial\tau$ (or by integration of (14), ϕ_s itself) at the surface. The lower boundary condition for shallow water is no normal flow at the bottom (zero normal derivative of ϕ), and for deep water vanishing flow at great depth (ϕ tending to zero).

In the case of rotational flow, the complementary problem arises. One has

$$u = \frac{\partial\psi}{\partial y} \tag{24}$$

$$v = \frac{\partial\psi}{\partial x} \tag{25}$$

$$\nabla^2\psi = -\omega, \tag{26}$$

where the vorticity ω is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{27}$$

and must either be carried as a variable of integration or specified by a turbulence model. Time advancement of (10) provides values

$$q_s = (1 + h'^2)^{1/2} (\mathbf{n} \cdot \nabla\psi)_s, \tag{28}$$

while (1) becomes

$$\frac{\partial h}{\partial t} = -(1 + h'^2)^{1/2} (\boldsymbol{\tau} \cdot \nabla\psi)_s = -\frac{\partial\psi_s}{\partial x}. \tag{29}$$

One recognizes the second equality in (29) as Stoker's statement of mass conservation [8]. Thus in the vorticity–streamfunction formulation, knowledge of surface values allows prediction of $\partial\psi/\partial n$. The boundary value problem is thus to find $\partial\psi/\partial\tau$, given $\partial\psi/\partial n$ at the surface. The lower boundary conditions are physically the same as in the irrotational case. Without loss of generality, they can be stated as $\psi = 0$ for shallow water and $\psi \to 0$ for deep water.

Summary

The governing equations for a fluid surface lead to a conservation law (10) for a surface velocity variable $q_s = u_s + h'v_s$. This "conservation-of-velocity" law is an extension of Kelvin's theorem to a fixed Eulerian frame of reference, and recovers Bernoulli's law for irrotational flow. The result (10) may be incorporated in a Boussinesq-type wave model, extending its accuracy to all orders in nonlinearity. Such a model has been successfully constructed and applied to irrotational waves in shallow water [1]. Deep water waves and rotational flows are also amenable to description by such a model with the addition of an appropriate solver for Poisson's equation and vorticity equations, respectively. The importance of the results derived here is that when the surface forcing **F** is specified, existing techniques for solving Poisson's equation in shallow water [1] or deep water [4, 5] allow the entire system to be solved in one spatial dimension.

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